

Forced Edges and Graph Structure ^{*}

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Abstract

For a degree sequence, we define the set of edges that appear in every labeled realization of that sequence as forced, while the edges that appear in none as forbidden. We examine structure of graphs whose degree sequences contain either forced or forbidden edges. Among the things we show, we determine the structure of the forced or forbidden edge sets, the relationship between the sizes of forced and forbidden sets for a sequence, and the resulting structural consequences to their realizations. This includes showing that the diameter of every realization of a degree sequence containing forced or forbidden edges is no greater than 3, and that these graphs are maximally edge-connected.

1 Basic Definitions and Results

We begin with some needed definitions and results. A *degree sequence* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a set of non-negative integers such that $n - 1 \geq \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 0$. The complement of a sequence α is the sequence $\bar{\alpha}$ where $\bar{\alpha}_i = n - \alpha_{n+1-i} - 1$. A sequence that corresponds to the vertex degrees of some simple graph is called a *graphic* degree sequence. A graph whose vertex degrees match a degree sequence is termed a *realization* of that sequence. To represent

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the degree sequence α of a given graph G , we will use the notation $\deg(G)$ where $\deg(G) = \alpha$.

For a realization $G = (V, E)$ of the sequence α , we will use the notation v_i to represent the vertex in G whose vertex degree corresponds to the i th value in the sequence α , while the neighborhood, or set of adjacent vertices, of v_i is represented as $N(v_i)$, i.e., $|N(v_i)| = \alpha_i$. For a subset of vertices $S \subseteq V$, the induced subgraph on this subset is represented as $G[S]$. An edge between two vertices, v_i and v_j , is designated as $v_i v_j$. For a graph G , we denote the diameter of the graph as $\text{diam}(G)$. An *alternating n -cycle* of a graph G is a configuration consisting of vertices v_1, \dots, v_n such that $\{v_1 v_2, v_3 v_4, \dots, v_{n-1} v_n\} \subseteq E$ and for each $v \in \{v_2 v_3, v_4 v_5, \dots, v_n v_1\}$, $v \notin E$.

For a graphic degree sequence α , we define its *forbidden set* $\mathcal{B}(\alpha)$ as the set of all edges between labeled nodes that do not appear in any realization of α , while its *forced set* $\mathcal{F}(\alpha)$ is the set of all edges between labeled nodes that appear in every realization of α . We will also define the set $\mathcal{P}(\alpha)$ to contain all the vertices that are in some forced edge in α , i.e., $\mathcal{P}(\alpha) = \{v_i | \exists v_j : v_i v_j \in \mathcal{F}(\alpha)\}$.

In order to compare degree sequences, we will use the following partial ordering of *majorization*. A degree sequence α majorizes (or dominates) the integer sequence β , denoted by $\alpha \succ \beta$, if for all k from 1 to n

$$\sum_{i=1}^k \alpha_i \geq \sum_{i=1}^k \beta_i, \quad (1)$$

and if the sums of the two sequences are equal. A convenient fact that we will use is that the majorization order is preserved by the complements of sequences, i.e., if $\alpha \succ \beta$ then $\bar{\alpha} \succ \bar{\beta}$.

For convenience, we introduce a notation for showing increments or decrements to specific indices in a sequence. For the degree sequence α , the sequences $\ominus_{i_1, \dots, i_k} \alpha$ and $\oplus_{i_1, \dots, i_k} \alpha$ are defined by

$$(\ominus_{i_1, \dots, i_k} \alpha)_i = \begin{cases} \alpha_i - 1 & \text{for } i \in \{i_1, \dots, i_k\} \\ \alpha_i & \text{otherwise,} \end{cases} \quad (2)$$

$$(\oplus_{i_1, \dots, i_k} \alpha)_i = \begin{cases} \alpha_i + 1 & \text{for } i \in \{i_1, \dots, i_k\} \\ \alpha_i & \text{otherwise.} \end{cases} \quad (3)$$

There is a straightforward but nontrivial relationship between majorization and the decrementing and incrementing operations.

Theorem 1 (Fulkerson and Ryser [1], Lemma 3.1). *If $\alpha \succ \beta$ and $\omega_1 = \{i_1, \dots, i_k\}$ and $\omega_2 = \{j_1, \dots, j_k\}$, where $i_1 \geq j_1, \dots, i_k \geq j_k$ then $\oplus_{\omega_1} \alpha \succ \oplus_{\omega_2} \beta$ and $\ominus_{\omega_2} \alpha \succ \ominus_{\omega_1} \beta$.*

For our purposes, the usefulness of comparing degree sequences using majorization stems from the following result.

Theorem 2 (Ruch and Gutman [2], Theorem 1). *If the degree sequence α is graphic and $\alpha \succ \beta$, then β is graphic.*

Finally, we will use another classic result.

Theorem 3 (Kleitman and Wang [3], Theorem 2.1). *For a degree sequence α and an index i , let α' be the sequence created by subtracting 1 from the first α_i values in α not including index i and then setting $\alpha_i = 0$. Then, the degree sequence α is graphic if and only if the degree sequence α' is graphic.*

2 Forced and Forbidden Edges

A simple observation about forced and forbidden edges set is that they have a dual relationship through their complement degree sequences.

Observation 4.

$$\mathcal{F}(\alpha) = \mathcal{B}(\bar{\alpha}).$$

A method for determining whether an edge is either forced or forbidden for a degree sequence is given by the next theorem.

Theorem 5. *Let α be a graphic degree sequence and $i, j \in \{1, \dots, n\}$ with $i \neq j$. The $v_i v_j \in \mathcal{F}(\alpha)$ if and only if*

1. $\oplus_{i,j} \alpha$ is not graphic,
2. for any realization G of α , $v_i v_j$ is not a element of any alternating n -cycle where n is even,
3. for any realization G of α and for every vertex set S , where $\{i, j\} \subseteq S \subseteq V$, $v_i v_j \in \mathcal{F}(\deg(G[S]))$,

Proof. The first part of this theorem was initially proved by Blitzstein and Diaconis ([4], Proposition 6.2), but it can be viewed as a consequence of Kundu's theorem [5]. If $\oplus_{i,j}\alpha$ is graphic, then Kundu's theorem guarantees that there exists a realization G of $\oplus_{i,j}\alpha$ containing the edge $v_i v_j$ as a $(0,1)$ -factor. Removing the edge $v_i v_j$ from G shows that it is not forced in α . The other direction is trivial.

If an alternating n -cycle exists containing the edge $v_i v_j$, then obviously that edge cannot be forced. The other direction follows from the result that every realization of a degree sequence can be transformed from any other realization by a series of edge switches of edges in an alternating 4-cycle in the graph [6, 7]. If an edge $v_i v_j$ is not forced, then there exists a sequence of edge switches through alternating 4-cycles that will eventually remove that edge. We take a minimal set of alternating 4-cycles to remove an edge $v_i v_j$ from a realization. Since the set is minimal each edge switch creates a $v_k v_l$ used by the next edge-switch. If edges $v_p v_q$ and $v_k v_l$ are in an alternating $(n-2)$ -cycle and edges $v_k v_l$ and $v_i v_j$ are in an alternating 4-cycle, then $v_p v_q$ and $v_i v_j$ are in an alternating n -cycle. Thus, by induction, we can merge the 4-cycles in the minimal set to form an alternating n -cycle containing $v_i v_j$.

For the third part, assume that the edge $v_i v_j$ is forced in $G = (V_G, E_G)$ but not in the induced subgraph $G[S]$. Take a realization $R = (V_R, E_R)$ of the degree sequence of $G[S]$ that does not contain the edge $v_i v_j$, and create a new graph $C = (V_G, E_C)$ where for any two vertices $v_p, v_q \in V_G$, if $v_p, v_q \in S$ then $v_p v_q \in E_C$ if and only if $v_p v_q \in E_R$; else, $v_p v_q \in E_C$ if and only if $v_p v_q \in E_G$. This graph C defines a realization of α that does not contain the edge $v_i v_j$ causing a contradiction. \square

There is a simple extension of Theorem 5 for sets of forbidden edges through the complement sequence. This includes a test for determining if an edge $v_i v_j$ is forbidden by testing whether or not $\ominus_{i,j}\alpha$ is graphic. By extending Theorem 5 with the Ruch and Gutman result, we also show an edge-inclusion result for forced and forbidden sets of a degree sequence.

Theorem 6. *For the graphic degree sequence α , if $v_i v_j \in \mathcal{F}(\alpha)$, then for all indices p, q where $1 \leq p \leq i$ and $1 \leq q \leq j$ and $p \neq q$, $v_p v_q \in \mathcal{F}(\alpha)$.*

Proof. Suppose that $v_p v_q$ is not forced, then Theorem 5 implies that $\oplus_{p,q}\alpha$ is graphic. From Theorem 1, it follows that $\oplus_{p,q}\alpha \succcurlyeq \oplus_{i,j}\alpha$, and Theorem 2 implies that $\oplus_{i,j}\alpha$ is graphic, contradicting the assumption that $v_i v_j$ is forced. \square

An immediate consequence of this proposition is that if there exist any forced edges for a degree sequence, then the edge $v_1 v_2$ must be one of them. This gives

linear-time methods to determine if a sequence has any forced or forbidden edges by testing whether $\oplus_{1,2}\alpha$ or $\ominus_{n-1,n}\alpha$ are graphic respectively.

Theorem 6 is also enough to establish the structure of the sets of forced and forbidden edges. While the induced subsets $S \subseteq \mathcal{P}(\alpha)$ do not necessarily need to be threshold graphs, the sets of forced edges for a degree sequence always do form a threshold graph.

Theorem 7. *For a graphic degree sequence α , the graph $G = (\mathcal{P}(\alpha), \mathcal{F}(\alpha))$ is a threshold graph.*

Proof. We want to show that the induced subgraph on any four vertices in G cannot be either $2K_2$, P_4 , or C_4 thus showing that the set of edges form a threshold graph [8]. Select any two edges in G having four unique vertices, $v_p v_q$ and $v_r v_s$. Since the vertices are unique, then we will assume without a loss of generality that $p < q$, $r < s$, and $p < r$. From Theorem 6, the edge $v_p v_r$ must also be forced so $2K_2$ cannot be induced. If $p < q < r < s$ or $p < r < q < s$, then Theorem 6 guarantees that the edge $v_r v_q$ is forced thus preventing C_4 and P_4 from being induced. Similarly, if $p < r < s < q$, then the edge $v_p v_s$ is forced for the same result, thus confirming the theorem. \square

Over the set of partitions for some positive integer p , majorization forms a lattice [9]. In these partition lattices, at the top of the graphic sequences are the threshold sequences in which every edge is forced. In contrast, regular sequences, which occupy the bottom of the lattice, cannot have any forced or forbidden edges (other than trivially with the complete or empty sequences). This follows from a strict ordering of forced and forbidden sets by subset down chains in this lattice.

Theorem 8. *For the graphic sequences α and β , if $\alpha \succcurlyeq \beta$ then $\mathcal{F}(\alpha) \supseteq \mathcal{F}(\beta)$ and $\mathcal{B}(\alpha) \supseteq \mathcal{B}(\beta)$.*

Proof. From the assumption $\alpha \succcurlyeq \beta$, it follows from Theorem 1 that $\oplus_{p,q}\alpha \succcurlyeq \oplus_{p,q}\beta$. If an edge $v_p v_q \notin \mathcal{F}(\alpha)$ then $\oplus_{p,q}\alpha$ is graphic and so $\oplus_{p,q}\beta$ must also be graphic. Thus $v_p v_q \notin \mathcal{F}(\beta)$ and so $\mathcal{F}(\alpha) \supseteq \mathcal{F}(\beta)$. The implication $\mathcal{B}(\alpha) \supseteq \mathcal{B}(\beta)$ immediately follows from the complement sequences. \square

3 Structure of Realizations

A useful result with structural implications is that forced (and forbidden) edges for a degree sequence also imply independent sets (or cliques) in realizations of the degree sequence.

Theorem 9. *Let α be a degree sequence.*

1. *If $v_i v_j \in \mathcal{F}(\alpha)$, then for any realization G of α , the set of vertices $V - (N(i) \cup N(j))$ forms an independent set,*
2. *If $v_i v_j \in \mathcal{B}(\alpha)$, then for any realization G of α , the set of vertices $N(i) \cup N(j)$ forms a clique.*

Proof. For the first statement, suppose there are vertices $\{v_p, v_q\} \subseteq V - (N(i) \cup N(j))$ that have edge between them. This forms an alternating 4-cycle in G , where we can replace the edges $\{v_p v_q, v_i v_j\}$ with $\{v_p v_i, v_q v_j\}$ forming a realization of α without the edge $v_i v_j$, causing a contraction. The second statement is immediate through the complement sequence of α . \square

Extending this results, we now relate the size of the sets of forced edges to forbidden edges for a degree sequence, by showing that forbidden edges in a degree sequence imply cliques of forced edges.

Theorem 10. *Let α be a graphic sequence where $\alpha_n > 0$. If $v_i v_j \in \mathcal{B}(\alpha)$ then there exists a clique of α_i nodes in $\mathcal{F}(\alpha)$.*

Proof. From Theorem 9, for any realization of α , the vertices $N(i) \cup N(j)$ form a clique. Using the Kleitman-Wang theorem, we construct a realization H of α where the first α_i vertices are connected to v_i , and so these first α_i vertices form a K_{α_i} -clique. For the degree sequence η created by removing the vertex v_i and its adjacent edges from H , this clique of the first α_i vertices must exist in every realization, i.e., $K_{\alpha_i} \subseteq \mathcal{F}(\eta)$.

Now take an arbitrary realization G of α . If we remove v_i along with its adjacent edges from G , then for the resulting graph G' with its degree sequence $\deg(G')$, it is straightforward to see that $\eta \preceq \deg(G')$. From Theorem 8, $\mathcal{F}(\eta) \subseteq \mathcal{F}(\deg(G'))$ and so any realization of $\deg(G')$ must contain all the edges in $\mathcal{F}(\eta)$, specifically K_{α_i} . By adding back the vertex v_i , we see that every realization G will also contain those edges. \square

We can extend this result to show forced cliques based on the minimum degree value.

Corollary 11. *For the graphic sequence α where $\alpha_1 < n - 2$ and $\mathcal{F}(\alpha) \neq \emptyset$, then $\mathcal{F}(\alpha)$ contains a clique of size α_n .*

Proof. Applying Theorem 10 to the complement sequence $\bar{\alpha}$, there must exist a forbidden set of size $n - 1 - \alpha_2$ in α . Since $\alpha_1 < n - 2$, then $|\mathcal{B}(\alpha)| \geq 2$. Thus α_n must be in a forbidden edge with α_{n-1} . Then applying Theorem 10 again, we arrive that $\mathcal{F}(\alpha)$ must contain a clique of size α_n . \square

We now show that having forced or forbidden edges for a degree sequence limits the diameter of its realizations.

Theorem 12. *For the graphic sequence α where $\alpha_n \geq 1$, if $\mathcal{F}(\alpha) \neq \emptyset$, or $\mathcal{B}(\alpha) \neq \emptyset$, then for any realization G of α ,*

$$\text{diam}(G) \leq 3. \quad (4)$$

Proof. We begin with a consideration of the case when $\mathcal{F}(\alpha) \neq \emptyset$. by partitioning the set of vertices of G into three sets where $V = \mathcal{P}(\alpha) \cup Q \cup R$. We define the set Q as the all the vertices in G that are adjacent to a vertex in $\mathcal{P}(\alpha)$, but are not themselves in $\mathcal{P}(\alpha)$. We next define the set R as all the remaining vertices, $R = V - \mathcal{P}(\alpha) - Q$. By performing a case analysis, we show that for any two vertices v_i and v_j in G , there is a path between them of length no greater than 3.

Case $\{v_i, v_j\} \subseteq \mathcal{P}(\alpha)$: Theorem 7 says that the forced edges between the vertices in $\mathcal{P}(\alpha)$ form a connected threshold graph, implying that the minimum path length between any two vertices in $\mathcal{P}(\alpha)$ is no more than 2.

Case $v_i \in Q, v_j \in \mathcal{P}(\alpha)$: From the definition of Q and Theorem 7, the path length between v_i and v_j is no more than 3.

Case $\{v_i, v_j\} \subseteq Q$: Let $v_k \in N(v_i)$ and $v_l \in N(v_j)$ where $\{v_k, v_l\} \subseteq \mathcal{P}(\alpha)$. If $v_k = v_l$ or the edge $v_k v_l \in E$, then we have found a path of length no more than 3 between v_i and v_j . Else, from Theorem 7 we can find a path with length 2 composed of forced edges from v_k to v_l ; let us assume that the path goes through v_m . If the edge $v_i v_m \notin E$ then there would exist an alternating 4-cycle where we could replace the edges $\{v_i v_k, v_l v_m\}$ with $\{v_i v_m, v_k v_l\}$ violating the assumption that $v_l v_m \in \mathcal{F}(\alpha)$. A similar argument establishes that $v_j v_m$ must also be in G , giving a path of length 2 from v_i to v_j through v_m .

Case $v_i \in R, v_j \in \mathcal{P}(\alpha)$: From Theorem 9, since $N(v_i) \subseteq Q$, then any vertex $v_k \in N(v_i)$ that we choose will be in Q . Now select two vertices $\{v_m, v_n\} \subseteq \mathcal{P}(\alpha)$ such that $v_m \in N(v_k)$ and $v_m v_n \in \mathcal{F}(\alpha)$. We first note that G also must contain the edge $v_k v_n$, because if $v_k v_n$ did not exist then G would have an alternating 4-cycle where we could replace the edges $\{v_i v_k, v_m v_n\}$ in G with the set $\{v_i v_m, v_k v_n\}$ violating the assumption that $v_m v_n \in \mathcal{F}(\alpha)$. Now because all the forced edges are connected, we can inductively extend this argument to show that every forced edge must be in a triangle with v_k . Thus v_i can reach any vertex $v_j \in \mathcal{P}(\alpha)$ with a path of length 2,

Case $v_i \in R, v_j \in Q$: The argument for proceeding case shows that v_i can reach any vertex in Q with a path of no more than length 3 by going through some vertex in $\mathcal{P}(\alpha)$.

Case $\{v_i, v_j\} \subseteq R$: Choose two vertices $v_k \in N(v_i)$ and $v_l \in N(v_j)$ and an edge $v_m v_n \in \mathcal{F}(\alpha)$. If $v_k = v_l$ then we found a path of length 2. If not then the edge $v_k v_l$ must be in E , or else there would exist an alternating 6-cycle where we could replace the edges in $\{v_i v_k, v_j v_l, v_m v_n\}$ with $\{v_k v_l, v_i v_m, v_j v_n\}$ violating the assumption that $v_m v_n \in \mathcal{F}(\alpha)$. Thus there is a path of no more than length 3 between v_i and v_j .

For the second part of the statement when $\alpha_n \geq 1$ and $\mathcal{B}(\alpha) \neq 0$, we note that Theorem 10 coupled with the proof of the first part of Theorem 12 is almost enough to prove the second part; it only fails when except when the forbidden edges are strictly between vertices of degree 1. To show the complete statement, assume that $v_m v_n \in \mathcal{B}(\alpha)$. Since v_m and v_n are not isolated, then we choose the vertices $v_p \in N(v_m)$ and $v_q \in N(v_n)$ where v_p and v_q are not necessarily distinct. Theorem 9 says that for every realization G of α , the vertices in $N(m) \cup N(n)$ form a clique. If the diameter of the graph is greater than 3, then there would have to exist a minimal 4-path in G between two vertices v_i and v_j . Without a loss of generality, we can assume that neither the vertex v_i nor its neighbor in that path v_k is in $N(m) \cup N(n)$, or else we could find a 3-path from v_i to v_j . But this would create an alternating 6-cycle where we could replace the edges in $\{v_i v_k, v_m v_p, v_n v_q\}$ with $\{v_m v_n, v_i v_p, v_k v_q\}$ violating the assumption that $v_m v_n \in \mathcal{B}(\alpha)$. \square

We now examine the edge connectivity of a graph whose degree sequence contains either a forced or forbidden edge. The edge connectivity $\lambda(G)$ is the minimum cardinality of an edge-cut over all edge-cuts of G . There is a trivial

upper bound for $\lambda(G) \leq \alpha_n$ where $\alpha = \deg(G)$. When a graph G has this edge connectivity of $\lambda(G) = \alpha_n$, then it is said to be maximally edge-connected. Any realization of a degree sequence with either forced or forbidden edges is maximally edge-connected.

Theorem 13. *For the graphic sequence α where $\alpha_n \geq 1$, if $\mathcal{B}(\alpha) \neq \emptyset$ or $\mathcal{F}(\alpha) \neq \emptyset$, then for any realization G of α ,*

$$\lambda(G) = \alpha_n. \quad (5)$$

Proof. We begin with some simple observations about what is required for a graph to be maximally edge-connected. If $\alpha_n = 1$, then for the connected graph G , $\lambda(G) = \alpha_n$ is trivially true; thus we assume that $\alpha_n \geq 2$. In addition, a result by Plesník [10] establishes that if $\text{diam}(G) \leq 2$, then G is maximally edge-connected. Thus, from Theorem 12, if G is not maximally edge-connected, then $\text{diam}(G) = 3$.

For a contradiction, we assume that there is a realization of G where $\lambda(G) < \alpha_n$. We denote the edge set S as an arbitrary minimum edge-cut of G , and the two components of G with S removed as P and Q . For each set P and Q , we partition each into two sets, $P = P_s \cup P_n$ (or $Q = Q_s \cup Q_n$), where P_s (or Q_s) is the set of vertices in P (or Q) with an adjacent edge in S , and P_n (or Q_n) are the remaining vertices.

Using an argument first presented by Hellwig and Volkmann [11], we show that $|P_n| \geq 2$. From the assumption that $\lambda(G) \leq \alpha_n - 1$, then

$$\alpha_n |P| \leq \sum_{p \in P} \deg(p) \leq |P|(|P| - 1) + \alpha_n - 1, \quad (6)$$

which implies that $|P| \geq \alpha_n + 1$. Along with the assumption that $|P_s| \leq \lambda(G) \leq \alpha_n - 1$, it follows that $|P_n| = |P| - |P_s| \geq 2$. There is a similar argument to show that $|Q_n| \geq 2$ also. One implication from this result is that since G is not maximally edge-connected, then $\alpha_1 \leq n - 3$.

We now show that if $v_i v_j \in \mathcal{F}(\alpha)$, then v_i and v_j must be in separate components. Suppose that $\{v_i, v_j\} \subseteq Q$, then from the proceeding argument, there must be at least one edge $v_k v_l$ strictly in P . This edge $v_k v_l$ would allow us to replace $\{v_i v_j, v_k v_l\}$ with $\{v_i v_k, v_j v_l\}$ violating the assumption that $v_i v_j$ is forced; thus, each forced edge must be in S . Extending this observation shows that if $K_3 \subseteq \mathcal{F}(\alpha)$, we would have a contradiction with G not being maximally edge-connected.

Let us consider the case where the forbidden edge set for α is not empty, $\mathcal{B}(\alpha) \neq \emptyset$. From Theorem 10, since $\alpha_n \geq 2$, then G has a clique of α_n in $\mathcal{F}(\alpha)$,

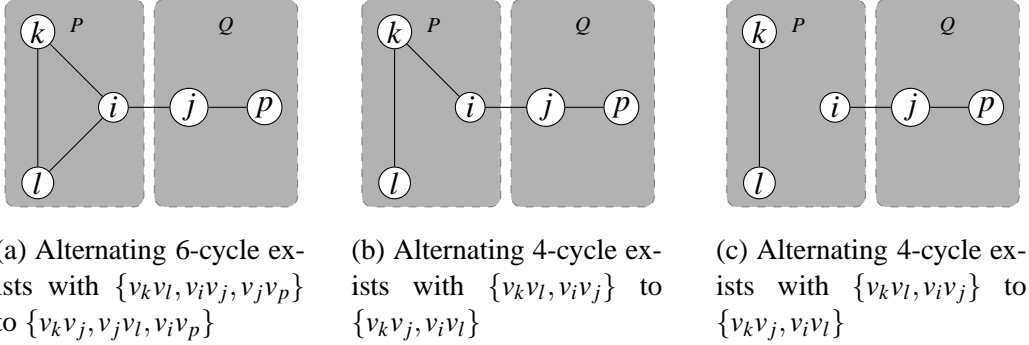


Figure 1: The three possible cases when $v_k v_l \in E$ and $\{k, l\} \subseteq P_n$. In all three cases, the edge $v_i v_j$ is not forced causing a contradiction.

and so if $\alpha_n > 3$, then $K_3 \in \mathcal{F}(\alpha)$ proving that G is not maximally edge-connected. Thus the only possible case for α not covered by this result is when $\alpha_n = 2$ and the resulting forced edge $v_i v_j$ makes up the set S . Assuming that $\{v_k, v_l\} \subseteq P_n$ and $v_p \in Q_n$, then if there would exist an edge between v_k and v_l the induced subgraph $G[\{v_i, v_j, v_k, v_l, v_p\}]$ would be one of the three cases in Figure 1. Since in all three cases the edge $v_i v_j$ is not forced, then the edge $v_k v_l$ cannot exist. This means that in general that any vertex in P_n (or Q_n) must be connected to members of P_s (or Q_s) only, and specifically, in this case, $\deg(v_k) = \deg(v_l) = 1$. This is a contradiction to $\alpha_n \geq 2$, and thus it follows that if $\mathcal{B}(\alpha) \neq \emptyset$, then $\lambda(G) = \alpha_n$.

When the forced edge set is not empty, we again use Theorem 10, this time on the complement sequence $\bar{\alpha}$, to show that forbidden edge set $\mathcal{B}(\alpha)$ has clique of size $n - 1 - \alpha_1$. Since $\alpha_1 \leq n - 3$, then the forbidden edge set is not empty, and thus to avoid a contradiction, then G must be maximally edge-connected. \square

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